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STAR-IN-COLORING OF COMPLETE BI-PARTITE GRAPHS, WHEEL GRAPHS AND PRISM GRAPHS

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ABSTRACT

A $k-$coloring of a graph $G = (V, E)$ is a mapping $c: V(G) \rightarrow \{1,2,3, \dots\}$ such that $uv \in E(G) \Rightarrow c(u) \neq c(v)$. In this paper, we have considered a complete bi-partite graph $K_{m,n}$ for all m, n and proved that the star-in-chromatic number of $K_{m,n}$ is either $n+1$ if $m \geq n$ or $m+1$ if $n > m$ respectively. We have also found that the star-in-chromatic number of a wheel graph W_n has the lower bound and upper bound as $4 \leq \chi_{si}(W_n) \leq 5$. Further we have considered the prism graph $Y_{n,m}$ and found that the star-in-chromatic number of the prism graph satisfies the relation $5 \leq \chi_{si}(Y_{n,m}) \leq 6$.

AMS Subject Classification: 05C15, 05C20

KEYWORDS: Complete Bi-Partite Graph, Wheel Graph, Prism Graph, Star-in-Coloring

INTRODUCTION

Graph coloring is one of the best known and popular research area in graph theory. Researchers have extensively worked on this field. Some conjectures are still open. Graph coloring has many applications and conjectures which are studied by various mathematicians and computer scientists around the world. The chromatic number and upper bounds of the chromatic number are still getting momentum. The notion of acyclic coloring of graphs was first introduced by Grünbaum[1] in 1973. Later he discussed the proper coloring avoiding 2-colored paths with four vertices as star-coloring in his paper. Star-colorings have recently been investigated by Fertin, et al[2] and Nešetřil, et al[3]. An orientation of a graph G is a directed graph obtained from G by choosing an orientation either from $u \rightarrow v$ or $v \rightarrow u$ for each edge $uv \in E(G)$. A proper coloring of a graph is an assignment of colors to the vertices such that adjacent vertices have different colors. The resultant graph is a digraph.

Definition 1

A *complete bi-partite graph* is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both end points in the same subset and each and every vertices in one vertex set is adjacent to all other vertices in another set. It is denoted by $K_{m,n}$ where m is the number of vertices in V_1 and n is the number of vertices in V_2. It consists of $m+n$ vertices and mn edges.

Definition 2

A *wheel graph* W_n is a graph with n vertices $n \geq 4$, formed by connecting a single vertex to all vertices of an $(n-1)$ cycle. It consists of n vertices and $2(n-1)$ edges.

Definition 3

A *prism graph* $Y_{n,m}$ is a simple graph obtained by the cartesian productof two graphs say cycle C_n and path P_m. It consists of nm vertices and $n(2m-1)$ edges.

Definition 4

A *star-coloring* of a graph G is a proper coloring of a graph with the condition that no path on four vertices (P_4) is 2-colored.

A k −star coloring of a graph G is a star coloring of G using atmost k colours.

Definition 5

An *in-coloring* of a digraph G is a proper coloring of the underlying graph G if for any path P_3 of length 2 with the end vertices of the same color, then the edges of $P_3$are oriented towards the central vertex.

By combining these two definitions, Sudha, et al [4] defined the star-in-coloring of graphs and is as follows:

Definition 6

A graph G is said to be *star-in-colored* if

- No path on four vertices is bicolored.
- Any path of length 2 with end vertices of same color are directed towards the middle vertex.

The minimum number of colors required to color the graph G satisfying the above conditions for star-in-coloring is called *star-in-chromatic number* of G and is denoted by $\chi_{si}(G)$.

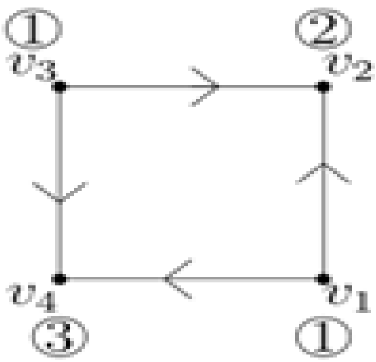

Figure 1: Star-in-Coloring of Cycle C_4

Figure 1 illustrates the star-in-coloring of cycle C_4. It consists of four vertices say $v_1, v_2, v_3, v_4$and four edges. The vertices v_1 and $v_3$are assigned the color say 1 and the vertices v_2 and $v_4$are assigned with colors 2 and 3 respectively.

Basic definitions and notations are from Harary[5].

STAR-IN-COLORING OF COMPLETE BI-PARTITE GRAPHS$K_{m,n}$

Theorem 1: The star-in-chromatic number of a complete bi-partite graph $k_{m,n}$ is either $n+1$ or $m+1$ accordingly $m \geq n$ or $n > m$ (for all m, n).

Poof: Consider a complete bi-partite graph $K_{m,n}$ which consists of $m+n$ vertices and mn edges. The complete bi-partite graph $K_{m,n}$ consists of two vertex sets: one vertex set is denoted by $\{v_1, v_2, \dots, v_m\}$ and the other vertex set is denoted by $\{u_1, u_2, \dots, u_n\}$. We need to show that the minimum number of colors required to color the graph is $n+1$ if $m \geq n$ and it is $m+1$ if $n > m$. To prove it let us define a function say c from the set of vertices to the set of natural numbers.

We define $c : V \to \{1,2,3,\dots\}$ such that $c(u) \neq c(v)$ if $uv \in E$ in $K_{m,n}$where V is the vertex set of $K_{m,n}$ and E is the edge set of $K_{m,n}$.

We define a general pattern of star-in-coloring for the vertices which require minimum number of colors to color the graph. It is defined as follows:

For $1 \leq i \leq m$

$$c(v_i) = \begin{cases} 1, & if\ m > n \\ 1, & if\ m = n \\ i+1, & if\ m < n \end{cases}$$

For $1 \leq j \leq n$

$$c(u_j) = \begin{cases} j+1, & if\ m > n \\ j+1, & if\ m = n \\ 1, & if\ m < n \end{cases}$$

Using the above general pattern the graph can be star-in-colored.

If $m > n$ then the star-in-chromatic number of $K_{m,n}$is $\chi_{si}(K_{m,n}) = n+1$;

if $m = n$ then the star-in-chromatic number of $K_{m,n}$ is $\chi_{si}(K_{m,n}) = n+1$;

if $m < n$ then the star-in-chromatic number of $K_{m,n}$ is $\chi_{si}(K_{m,n}) = m+1$.

Illustration 1

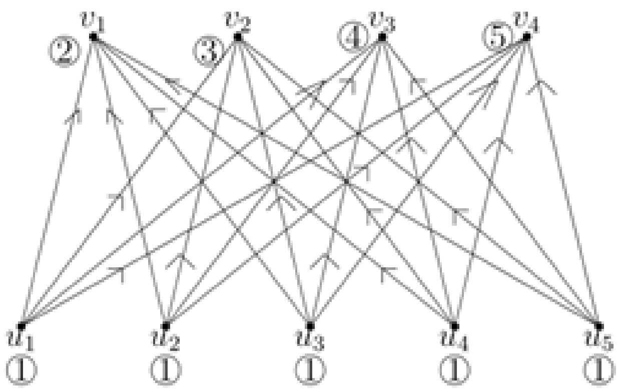

Figure 2: Complete Bipartite Graph, $K_{4,5}$

By using the above theorem, the graph $K_{4,5}$ is star-in-colored as shown in figure 2. The star-in-chromatic number of $K_{4,5}$ is $\chi_{si}(K_{4,5}) = 5$ ($Here\ m+1 = 4+1$, since $m < n$). All the edges in a path P_3 is oriented towards the center vertex if the end vertices of the path P_3have the same color.

Illustration 2

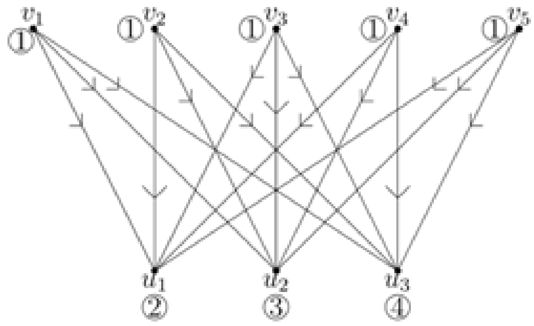

Figure 3: Complete Bipartite Graph, $K^{5,3}$

The star-in chromatic number of $K_{5,3}$ is $\chi_{si}(K_{5,3}) = 4$ $(Here\ n + 1 = 3 + 1$, since $m > n)$.

STAR-IN-COLORING OF WHEEL GRAPH W_n (n odd)

Theorem 2: The star-in-chromatic number of the wheel graph satisfies the inequality $4 \leq \chi_{si}(W_n) \leq 5$.

Proof: Consider a wheel graph W_n which consists of n vertices and $2(n-1)$ edges. The vertex in the center is denoted by v_0 and the other vertices are denoted by $v_1, v_2, \dots, v_{n-1}$. As per the definition of star-in-coloring no two adjacent vertices must receive the same color. Vertices in any path of length 3 must be colored with minimum of 3 colors and edges in any path of length two with end vertices having same color must be oriented towards the center vertex in such a way that each and every edge must be considered for in-coloring.

We define $c : V \to \{1,2,3,\dots\}$ such that $c(u) \neq c(v)$ if $uv \in E$ in W_n, where V is the vertex set in W_n and E is the edge set in W_n.

Case (i): For $n \equiv 1(mod\ 4), n > 5$

$$c(v_0) = 4$$

$$c(v_i) = \begin{cases} 1, \text{if i} \equiv 1(\text{mod } 2) \\ 2, \text{if i} \equiv 2(\text{mod } 4) \\ 3, \text{if i} \equiv 0(\text{mod } 4) \end{cases}$$

Case (ii): For $n \equiv 3(mod\ 4), n > 11$

$$c(v_0) = 4$$

$$c(v_i) = \begin{cases} 1, & if\ \text{i} \equiv 1(\text{mod } 2) \\ 2, & if\ \text{i} \equiv 2(\text{mod } 4)\ and\ i < n-1 \\ 3, & if\ \text{i} \equiv 0(\text{mod } 4)\ and\ i > 4 \\ 5, & if\ i = 4\ and\ i = n-1 \end{cases}$$

By using the above pattern, W_n for odd n admit star-in-coloring.

Remark: We have the general pattern of star-in-coloring only for odd n and not for even n. Since, if n is odd, in-coloring is satisfied if we assign a color say 1 to alternate vertices $v_1, v_3, \dots, v_{n-2}$ so that no two adjacent vertices have same color and if we assign any color to all other vertices then in-coloring is satisfied and later star-coloring is also satisfied by using the above pattern of coloring. If n is even, then two adjacent vertices say v_1 and v_{n-1} receive the same color if we color the vertices alternatively by using color 1 and in-coloring is not satisfied. Therefore W_n for even n does not admit star-in-coloring.

Illustration 3

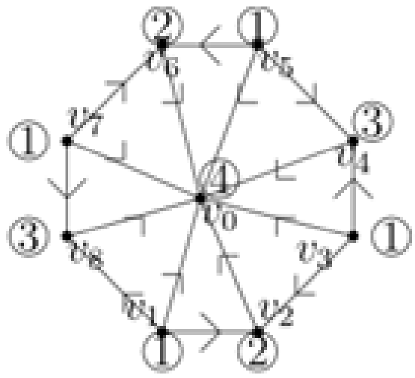

Figure 4: Wheel Graph W_9

By using the case (i) of the above theorem, the graph W_9 is star-in-colored as shown in figure 3. The star-in-chromatic number of W_9 is $\chi_{si}(W_n) = 4$.

Illustration 4

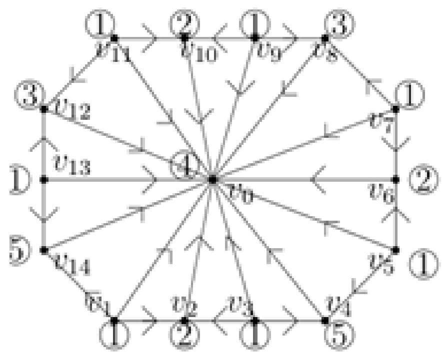

Figure 5: Wheel Graph W_{15}

The star-in-chromatic number of W_{15} is $\chi_{si}(W_{15}) = 5$ as per case (ii).

STAR-IN-COLORING OF PRISM GRAPHS $Y_{n,m}$

Theorem 3: The star-in-chromatic number of the prism graph is $5 \leq \chi_{si}(Y_{n,m}) \leq 6$.

Proof: Consider a prism graph $Y_{n,m}$which has mn vertices and $n(2m-1)$ edges. We consider the prism graph only for even n.

We define $c : V \to \{1,2,3,\dots\}$ such that $c(u) \neq c(v)$ if $uv \in E$ in $Y_{n,m}$, where V is the vertex set in $Y_{n,m}$ and E is the edge set in $Y_{n,m}$.

The vertex set V in $Y_{n,m}$ are partitioned into m vertex sets denoted by $V^1, V^2, \dots, V^m$ where each vertex set consists of n vertices. The vertex set V^jconsists of the vertices $v_1^j, v_2^j, \dots, v_n^j$ for all $1 \leq j \leq m$.

The general pattern of coloring has been grouped into two cases:

One for $n \equiv 0(mod\ 4)$ and other for $n \equiv 2(mod\ 4)$.

Case (i): For $n \equiv 0(mod\ 4)$, there are four cases:

Case (a): For j$\equiv 1(mod\ 4)$

$$c(v_i) = \begin{cases} 1, \text{if i } \equiv \ 1(\text{mod } 2) \\ 2, \text{if i } \equiv \ 2(\text{mod } 4) \\ 3, \text{if i } \equiv \ 0(\text{mod } 4) \end{cases}$$

Case (b): For j$\equiv 2(mod\ 4)$

$$c(v_i) = \begin{cases} 1, \text{if i } \equiv \ 0(\text{mod } 2) \\ 4, \text{if i } \equiv \ 1(\text{mod } 4) \\ 5, \text{if i } \equiv \ 3(\text{mod } 4) \end{cases}$$

Case (c): For j$\equiv 3(mod\ 4)$

$$c(v_i) = \begin{cases} 1, \text{if i } \equiv \ 1(\text{mod } 2) \\ 3, \text{if i } \equiv \ 2(\text{mod } 4) \\ 2, \text{if i } \equiv \ 0(\text{mod } 4) \end{cases}$$

Case (d): For j$\equiv 0(mod\ 4)$

$$c(v_i) = \begin{cases} 1, \text{if i } \equiv \ 0(\text{mod } 2) \\ 5, \text{if i } \equiv \ 1(\text{mod } 4) \\ 4, \text{if i } \equiv \ 3(\text{mod } 4) \end{cases}$$

According to case (i) the star-in-chromatic number of $Y_{n,m}$, $\chi_{si}(Y_{n,m})$ is 5.

Case (ii): For $n \equiv 2(mod\ 4)$

$$c(v_i^j) = 1, if\ i + j\ is\ even$$

For all other values of i and j, we consider four cases as follows:

Case (a): For $j \equiv 1(mod\ 4)$

$$c(v_i^j) = \begin{cases} 2, if i \equiv 2(mod\ 4) \\ 3, if\ i \equiv 0(mod\ 4) \end{cases}$$

Case (b): For $j \equiv 2(mod\ 4)$

$$c(v_i^j) = \begin{cases} 5, if\ i \equiv 1(mod\ 4) \\ 4, if\ i \equiv 3(mod\ 4) \end{cases}$$

Case (c): For $j \equiv 3(mod\ 4)$

$$c(v_i^j) = \begin{cases} 3, if\ i \equiv 2(mod\ 4) \\ 2, if\ i \equiv 0(mod\ 4) \end{cases}$$

Case (d): For $j \equiv 0(mod\ 4)$

$$c(v_i^j) = \begin{cases} 4, if\ i \equiv 1(mod\ 4) \\ 5, if\ i \equiv 3(mod\ 4) \end{cases}$$

In addition to this general pattern, we require the change of coloring in vertices as explained below:

The vertex v_n^1 takes the color 4, the vertex v_1^2 takes the color 3, the vertex v_2^3takes the color 5 and the vertex v_n^3 takes the color 6. Further, when $j = 4$ the vertices v_1^4, v_3^4 takes the color 2 and 3 respectively. When $j = 5$ the vertices $v_n^5,\ v_2^5, v_4^5$ takes the color 5, 4, 5 respectively. Similarly, this change of coloring pattern extends on the beginning and the ending vertices of each cycles till the j^{th} level where the color 6 appears. Other vertices takes the color according the above cases. Then this change of coloring pattern contracts in a similar way on the beginning and the ending vertices of each cycle still the j^{th} level where the color 6 appears. This process is continued. We need only 6 colors to color the graph. For example, the vertices v_5^{n+2} take the value 6 when $n \equiv 2(mod\ 4)$ for all n in general.

According to case (ii) the star-in-chromatic number of $Y_{n,m}$, $\chi_{si}(Y_{n,m})$ is 6.

With this pattern of coloring the prism graph $Y_{n,m}$ is star-in-colored.

Illustration 5

Consider the prism graph $Y_{8,4}(n = 8 \equiv 0(mod\ 4))$

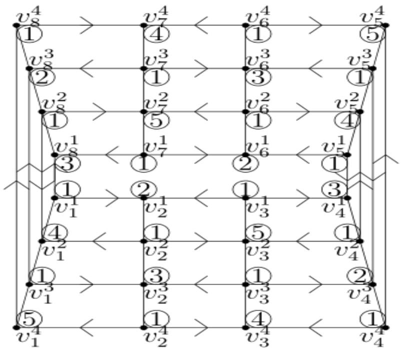

Figure 6: Prism Graph $Y_{8,4}$

By using the case (i) of the above theorem the graph $Y_{8,4}$ is star-in-colored as shown in figure 6. The star-in-chromatic number of $Y_{8,4}$, $\chi_{si}(Y_{8,4})$ is 5.

In a similar way, the prism graph for $n \equiv 2(mod\ 4)$ can be colored using case (ii) of the theorem-3.

CONCLUSIONS

In this paper, we have found that

- The star-in-chromatic number of complete bi-partite graphs is either $n + 1$ or $m + 1$ if $m \geq n$ or $n > m$ accordingly.
- The star-in chromatic number of the wheel graphs W_n for odd n satisfies $4 \leq \chi_{si}(W_n) \leq 5$.
- The star-in chromatic number of the prism graphs satisfies the inequality $5 \leq \chi_{si}(Y_{n,m}) \leq 6$.

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